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ON A TWO-DIMENSIONAL LAMB'S PROBLEM IN CASE OF POROELASTICITY

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Abstract. A two-dimensional boundary-value problem for a porous half-space with a permeable boundary, described by the governing Biot's equations of poroelasticity, is considered. Using complex analysis techniques, explicit formulas of a general solution are represented as a superposition of contributions from the different types (volume and surface waves) of motion.

Keywords: *Lamb's problem, poroelasticity, Biot's model, wave propagation, head wave, Rayleigh wave*.

AMS Subject Classification: *74A10, 74B05.*

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1. Introduction

We consider the initial boundary value problem of wave propagation in the half-plane $\mathsf{R}^2_+ = \{-\infty < x < +\infty,\ y \geq 0\}$ filled with a homogeneous, isotropic, porous medium saturated with a fluid, i.e., a Biot medium, which is at rest for $t < 0$. It is assumed that the pores of the boundary $y = 0$ are open. In this medium, the wave field is formed at $t = 0$ as the result of an external point action applied to the elastic phase (the skeleton of the medium), which is free of stresses and the pressure, of the boundary $y = 0$. This problem (non-stationary case) has been studied by L.A. Molotkov in (Molotkov, 2001), where the author has obtained formulas for components of displacement vectors in integral form. In (Gerasik & Stastna, 2008) V. Gerasik and M. Stastna have obtained the solution of the problem in the stationary case in the form of integrals the asymptotics of which, similarly to (Molotkov, 2001), was found subsequently with the help of the saddle-point method. In the present paper, formulae for the components of displacement vectors are obtained in explicit form. This paper is an extended version of the paper (Zavorokhin, 2013), where the main results have been announced. We are making this version available in order to have more clear results and discussions in comparison to its short version.

2. Statement of the problem. Representation of the solution via quadratures

The problem of finding perturbations in a Biot medium is reduced to the solution of wave equations (1) for potentials and to the determination of displacements and stresses by the following formulas (2), (3), and (4):

$$\left(\Delta-\frac{1}{v_i^2}\frac{\partial^2}{\partial t^2}\right)\varphi_i=0\ (i=1,2),\quad \left(\Delta-\frac{1}{v_3^2}\frac{\partial^2}{\partial t^2}\right)\boldsymbol{\psi}=0, \tag{1}$$

$$\mathbf{u}=\nabla\varphi_1+\nabla\varphi_2+\mathbf{rot}\boldsymbol{\psi}, \tag{2}$$

$$\mathbf{w}=B_1\nabla\varphi_1+B_2\nabla\varphi_2-\frac{\rho_f}{m}\mathbf{rot}\boldsymbol{\psi}, \tag{3}$$

$$\tau_{yy}=(\rho+B_1\rho_f)\frac{\partial^2\varphi_1}{\partial t^2}+(\rho+B_2\rho_f)\frac{\partial^2\varphi_2}{\partial t^2}-2L^{'}\left(\frac{\partial^2\varphi_1}{\partial x^2}+\frac{\partial^2\varphi_2}{\partial x^2}-\frac{\partial^2\psi_z}{\partial x\partial y}\right),$$

$$\tau_{xy}=L^{'}\left(2\frac{\partial^2\varphi_1}{\partial x\partial y}+2\frac{\partial^2\varphi_2}{\partial x\partial y}+\frac{\partial^2\psi_z}{\partial x^2}-\frac{\partial^2\psi_z}{\partial y^2}\right), \tag{4}$$

$$-p=(\rho_f+B_1 m)\frac{\partial^2\varphi_1}{\partial t^2}+(\rho_f+B_2 m)\frac{\partial^2\varphi_2}{\partial t^2}.$$

Here, $\varphi_i(x,y,t)$ and $\boldsymbol{\psi}(x,y,t)=\psi_z(x,y,t)\mathbf{k}$ are scalar and vector potentials describing two longitudinal waves P_i and a transverse wave S, which propagate with velocities $v_i, v_3=const>0, (i=1,2)$, respectively; $\mathbf{u}=(u_x,u_y)$ is the displacement vector in the elastic phase and $\mathbf{w}=\varepsilon(\mathbf{U}-\mathbf{u})=(w_x,w_y)$ is the displacement vector of fluid particles inside the pores relative to the skeleton; $\tau_{ij}=\sigma_{ij}+\Sigma_{ij}$ is the stress tensor in the porous medium; $\Sigma_{ij}=-\varepsilon p\delta_{ij}$, where p is the pressure in the fluid medium, δ_{ij} is the Kronecker symbol; ε is the porosity, ρ_f is the density of fluid, m is a parameter with dimension of density, ρ is the mean density of the porous medium and $L^{'}$ is the shear modulus of the porous medium – they are positive constants; $B_1, B_2=const$ are coefficients dependent on the structure of the porous medium.

Since the vector $\boldsymbol{\psi}$ is not uniquely defined, we set the additional condition $\mathrm{d}iv\boldsymbol{\psi}=0$.

The wave field arisen must satisfy for $y=0$ the boundary conditions

$$\tau_{yy}|_{y=+0}=-A\delta(x)\delta(t),\ \tau_{xy}|_{y=+0}=0,\ p|_{y=+0}=0,\ A=const>0 \tag{5}$$

and the zero initial data

$$\varphi_1=\varphi_2=\psi_z=0,\quad \frac{\partial\varphi_1}{\partial t}=\frac{\partial\varphi_2}{\partial t}=\frac{\partial\psi_z}{\partial t}=0\quad at\quad t=0. \tag{6}$$

The problem of finding perturbations in the Biot medium reduces to the solution of wave equations for potentials (1) and to determine the displacements and stresses according to formulas (2), (3), (4). With the help of the integral Fourier (with respect to the variable x) and Laplace (with respect to the variable t) transforms, the potentials as solutions of (1) can be represented by the formulas

$$\varphi_i=\int_0^\infty\frac{\cos kxdk}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}X_i(k,\eta)e^{k(t\eta-y\alpha_i)}d\eta\ (i=1,2), \tag{7}$$

$$\psi_z=\int_0^\infty\frac{\sin kxdk}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}Y(k,\eta)e^{k(t\eta-y\beta)}d\eta, \tag{8}$$

where

$$\sigma > 0, \alpha_i(\eta) = \sqrt{1+\frac{\eta^2}{v_i^2}}, \beta(\eta) = \sqrt{1+\frac{\eta^2}{v_3^2}} \ (i=1,2). \qquad (9)$$

For uniqueness of the radicals α_i, β on the plane η we draw cuts from the branch points $\eta = \pm i v_j$ $(j=1,2,3)$ into the left half-plane parallel to the real axis and fix the principal sheet by the conditions

$$\alpha_i = 1, \ \beta = 1 \quad \text{for } \eta = 0. \qquad (10)$$

In view of the choice of the principal sheet (10), for large η the following relations hold:

$$\alpha_i = \frac{\eta}{v_i}\left(1+O\left(\frac{v_i^2}{\eta^2}\right)\right) (i=1,2), \qquad (11)$$

$$\beta = \frac{\eta}{v_3}\left(1+O\left(\frac{v_3^2}{\eta^2}\right)\right). \qquad (12)$$

The functions $X_1(k,\eta), X_2(k,\eta), Y(k,\eta)$ are determined from the boundary conditions (5), which are reduced to the system of equations

$$\begin{gathered} g(X_1+X_2) - 2\beta Y = -\frac{1}{\pi L' k}, \\ 2\alpha_1 X_1 + 2\alpha_2 X_2 - gY = 0, \\ (v_2^2 - v_4^2)X_1 + (v_1^2 - v_4^2)X_2 = 0, \end{gathered} \qquad (13)$$

moreover, $g(\eta) = 2 + \frac{\eta^2}{v_3^2}, v_4 = \sqrt{\frac{\mathsf{M}}{m}} = const > 0$ is a parameter with dimension of velocity, and M is the modulus of the porous medium.

The solution of system (13) is represented by the relations

$$X_1 = -\frac{(v_1^2 - v_4^2)g}{\pi k L' \Delta_0}, \ X_2 = \frac{(v_2^2 - v_4^2)g}{\pi k L' \Delta_0},$$

$$Y = -\frac{2}{\pi L' k \Delta_0}(\alpha_1(v_1^2 - v_4^2) + \alpha_2(v_4^2 - v_2^2)), \qquad (14)$$

where

$$\Delta_0(\eta) = (v_1^2 - v_2^2)g^2 - 4\alpha_1\beta(v_1^2 - v_4^2) - 4\alpha_2\beta(v_4^2 - v_2^2). \qquad (15)$$

The equation $\Delta_0(\eta) = 0$ is the dispersion relation of surface Rayleigh waves for the free boundary of the porous Biot medium.

Substituting formulas (14) in (7), (8), and using relations (2), (3), we obtain expressions for displacements in the elastic phase and for relative displacements in the fluid phase:

$$u_x = \frac{A}{\pi L'}\int_0^{+\infty} \frac{\sin kx}{2\pi i} dk \int_{\sigma - i\infty}^{\sigma + i\infty} \Big[(v_1^2 - v_4^2)(ge^{-ky\alpha_1} - 2\alpha_1\beta e^{-ky\beta})$$

$$+ (v_4^2 - v_2^2)(ge^{-ky\alpha_2} - 2\alpha_2\beta e^{-ky\beta}) \Big] \frac{e^{kt\eta}}{\Delta_0} d\eta, \qquad (16)$$

$$u_y = \frac{A}{\pi L^{'}}\int_0^{+\infty}\frac{\cos kx}{2\pi i}dk\int_{\sigma-i\infty}^{\sigma+i\infty}\Big[(v_1^2-v_4^2)\alpha_1(ge^{-ky\alpha_1}-2e^{-ky\beta})$$

$$+(v_4^2-v_2^2)\alpha_2(ge^{-ky\alpha_2}-2e^{-ky\beta})\Big]\frac{e^{kt\eta}}{\Delta_0}d\eta, \tag{17}$$

$$w_x = \frac{A}{\pi L^{'}}\int_0^{+\infty}\frac{\sin kx}{2\pi i}dk\int_{\sigma-i\infty}^{\sigma+i\infty}\Bigg[(v_1^2-v_4^2)(B_1ge^{-ky\alpha_1}+\frac{2\alpha_1\beta\rho_f}{m}e^{-ky\beta})$$

$$+(v_4^2-v_2^2)(B_2ge^{-ky\alpha_2}+\frac{2\alpha_2\beta\rho_f}{m}e^{-ky\beta})\Bigg]\frac{e^{kt\eta}}{\Delta_0}d\eta, \tag{18}$$

$$w_y = \frac{A}{\pi L^{'}}\int_0^{+\infty}\frac{\cos kx}{2\pi i}dk\int_{\sigma-i\infty}^{\sigma+i\infty}\Bigg[(v_1^2-v_4^2)\alpha_1(B_1ge^{-ky\alpha_1}+\frac{2\rho_f}{m}e^{-ky\beta})$$

$$+(v_4^2-v_2^2)\alpha_2(B_2ge^{-ky\alpha_2}+\frac{2\rho_f}{m}e^{-ky\beta})\Bigg]\frac{e^{kt\eta}}{\Delta_0}d\eta. \tag{19}$$

3. Derivation of exact formulas for the solution

Our goal is to obtain the solution of the problem stated in explicit form. Following the method suggested by G.I. Petrashen in (Petrashen *et al.*,1950), we are able to integrate the expressions for displacements (16)-(19). This can be accomplished owing to the possibility of changing the order of integration with respect to η and k in (16) – (19), which is analyzed in papers (Petrashen *et al.*,1950; Babich *et al.*, 2002).

For definiteness, we consider the horizontal component of the displacement in the elastic phase u_x. Formulas for u_y, w_x, w_y are obtained similarly:

$$u_x = \frac{A}{\pi L^{'}}\int_0^{+\infty}\frac{\sin kx}{2\pi i}dk\int_{\sigma-i\infty}^{\sigma+i\infty}\Big[(v_1^2-v_4^2)(ge^{-ky\alpha_1}-2\alpha_1\beta e^{-ky\beta})$$

$$+(v_4^2-v_2^2)(ge^{-ky\alpha_2}-2\alpha_2\beta e^{-ky\beta})\Big]\frac{e^{kt\eta}}{\Delta_0}d\eta$$

$$=\frac{A}{\pi L^{'}}\int_0^{+\infty}\frac{\sin kx}{2\pi i}dk\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{(v_1^2-v_4^2)ge^{k(-\alpha_1 y+t\eta})}{\Delta_0}d\eta$$

$$+\frac{A}{\pi L^{'}}\int_0^{+\infty}\frac{\sin kx}{2\pi i}dk\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{(v_4^2-v_2^2)ge^{k(-\alpha_2 y+t\eta})}{\Delta_0}d\eta$$

$$-\frac{2A}{\pi L^{'}}\int_0^{+\infty}\frac{\sin kx}{2\pi i}dk\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{\beta(\alpha_1(v_1^2-v_4^2)+\alpha_2(v_4^2-v_2^2))e^{k(-\beta y+t\eta)}}{\Delta_0}d\eta$$

$$=u_{xp_1}+u_{xp_2}+u_{xs}. \tag{20}$$

Here, u_{xp_1}, u_{xp_2}, and u_{xs} contain the factors $e^{k(-\gamma y+t\eta)}, \gamma=\{\alpha_i,\beta\}\,(i=1,2)$. It is obvious that when $\mathrm{Re}(-\gamma y+t\eta)>0$, one cannot change the order of integration in (20), because we arrive at exponentially divergent integrals. Consequently, to accomplish this change, the contour of integration $(\sigma-i\infty,\sigma+i\infty)$ should be deformed in such a way that these expressions remain negative on it. Note that in $u_{xp_1}, u_{xp_2}, u_{xs}$ the integrands are regular for $\mathrm{Re}\eta > 0$, because the equation $\Delta_0(\eta) = 0$ does not have roots in the right half-plane on the principal sheet η. The only poles on the imaginary axis of the plane η may be the poles $\eta = \pm i v_R$ (v_R is the velocity of the Rayleigh wave) that coincide with the roots of the equation $\Delta_0(\eta) = 0$.

Under the conditions

$$v_2 > v_3 \quad or \quad v_3 > v_2, \left(2 - \frac{v_2^2}{v_3^2}\right)^2 \sqrt{1 - \frac{v_2^2}{v_1^2}} > 4\left(1 - \frac{v_4^2}{v_1^2}\right)\sqrt{1 - \frac{v_2^2}{v_3^2}};$$

the surface Rayleigh wave propagates along the free boundary $y = 0$ of the porous medium, but when

$$v_3 > v_2, \left(2 - \frac{v_2^2}{v_3^2}\right)^2 \sqrt{1 - \frac{v_2^2}{v_1^2}} < 4\left(1 - \frac{v_4^2}{v_1^2}\right)\sqrt{1 - \frac{v_2^2}{v_3^2}};$$

the Rayleigh wave is lacking (see Molotkov, L.A. (2001)).

Proposition 1. *If* $t < \dfrac{y}{v_i}$, *then* $u_{xp_i} = 0 (i = 1,2)$, *and if* $t < \dfrac{y}{v_3}$, *then* $u_{xs} = 0$.

Physical interpretation: the waves travel with the velocities v_j, $j = 1,2,3$, respectively, and do not reach the points with such y coordinate for the time t.

We consider

$$u_{xp_1} = \frac{A}{\pi L^{'}} \int_0^{+\infty} \frac{\sin kx}{2\pi i} dk \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{(v_1^2 - v_4^2) g e^{k(-\alpha_1 y + t\eta)}}{\Delta_0} d\eta. \tag{21}$$

For large η the integrand in (21) has the form

$$\frac{(v_1^2 - v_4^2) g e^{k\eta\left(t - \frac{y}{v_1}\right) + O\left(\frac{1}{\eta}\right)}}{\Delta_0}. \tag{22}$$

Deforming the contour of integration $(\sigma - i\infty, \sigma + i\infty)$ into a semicircle adjacent to the imaginary axis η, we obtain that the integral with respect to η in (21) equals zero in virtue of Jordan's lemma. Proof of the proposition (3.1)for u_{xp_2}, u_{xs} is the same.

Further, for $t > \dfrac{y}{v_1}$, using the formula $\sin kx = \dfrac{e^{ikx} - e^{-ikx}}{2i}$, we split the integral in (21) into two parts. Let us consider one of them

$$\frac{A}{4\pi^2 L^{'}} \int_0^{+\infty} dk \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{(v_1^2 - v_4^2) g e^{k(-ix - \alpha_1 y + t\eta)}}{\Delta_0} d\eta. \tag{23}$$

We study the behavior of the exponent on the imaginary axis η in the formula (23). We set $\eta = i\xi$ and denote by

$$\Psi(\eta) = i\Upsilon(\xi) = i\left(-x + t\xi - y\sqrt{\left(\frac{\xi}{v_1}\right)^2 - 1}\right). \tag{24}$$

We are interested in the behavior of the function $\Upsilon(\xi)$ for $\xi \geq v_1$, specifically the roots of its derivative with respect to ξ

$$\frac{d}{d\xi}\Upsilon(\xi) = t - \frac{y}{v_1}\frac{\frac{\xi}{v_1}}{\sqrt{\left(\frac{\xi}{v_1}\right)^2 - 1}}.$$

As $\xi \to +\infty$, then $\frac{d}{d\xi}\Upsilon(\xi) > 0$. As $\xi \to v_1 + 0$, then $\frac{d}{d\xi}\Upsilon(\xi) \to -\infty$. Therefore, $\frac{d}{d\xi}\Upsilon(\xi)$ has a root, and the ratio $\frac{\frac{\xi}{v_1}}{\sqrt{\left(\frac{\xi}{v_1}\right)^2 - 1}}$ is a monotonic function on the interval under consideration, and hence there is a unique root, which we denote as ξ_{st}. Note that

$$\frac{d^2}{d\xi^2}\Upsilon(\xi_{st}) = \frac{y}{v_1^2\left(\left(\frac{\xi_{st}}{v_1}\right)^2 - 1\right)^{\frac{3}{2}}} > 0. \tag{25}$$

The latter means that ξ_{st} is a minimum point of the function $\Upsilon(\xi)$, and $\Psi^{'}(\eta_{st}) = 0$, where $\eta_{st} = i\xi_{st}$.

We deform the contour into (23) so that $\mathrm{Re}\Psi(\eta) < 0$ is executed on it (see Fig. 1). Now changing the order of integration with respect to η and k in the formula (23) is possible. Integrals over k are easily calculated and we get

$$u_{xp_1} = -\frac{1}{4\pi^2 L^{'}}\int_{C^1}\frac{(v_1^2 - v_4^2)g}{\Delta_0}\left(\frac{1}{\Psi_-^1} - \frac{1}{\Psi_+^1}\right)d\eta, \tag{26}$$

where

$$\Psi_\pm^1 = \pm ix + t\eta - \alpha_1 y, \tag{27}$$

and C^1 is the contour shown in Fig. 1.

We complete the contour C^1 with a semicircle in the right half-plane, extended to the contour itself, and calculate the increment $arg\Psi(\eta)$ on this contour. It is easy to establish that somewhere inside the closed contour $\Psi(\eta)$ has exactly one root:

$$\eta_0^1 = \frac{ixt + y\sqrt{t^2 - \frac{x^2+y^2}{v_1^2}}}{t^2 - \frac{y^2}{v_1^2}}. \tag{28}$$

Note also that if $\Psi_-^1(\eta_0^1) = 0$, then $\Psi_+^1(\overline{\eta_0^1}) = 0$.

Now we can use the residue theorem:

- We have a closed contour.
- The integral over the semicircle tends to zero as its radius tends to infinity.

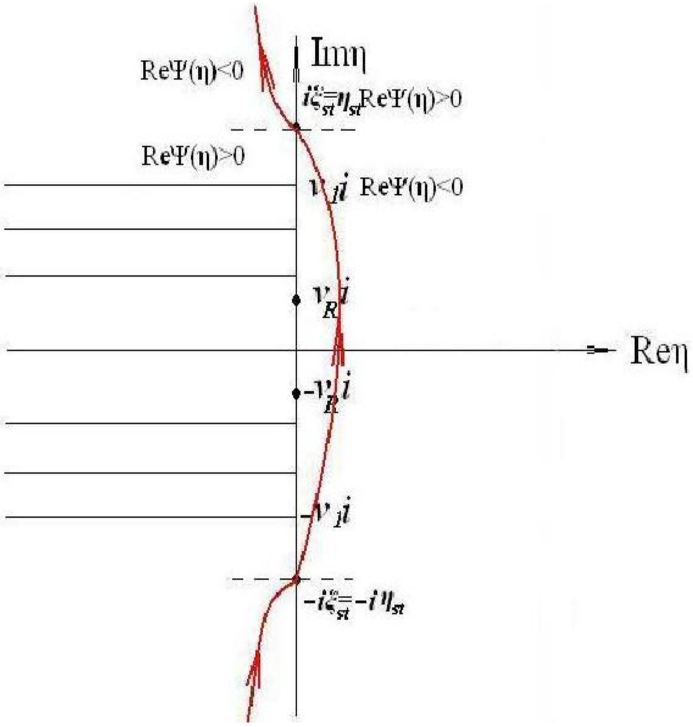

Figure 1. Integration contour C^1.

Thus, the desired integral in (26) is equal to $2\pi i$ multiplied by the residue with the opposite sign (due to the orientation of the closed contour in the counterclockwise direction):

$$u_{xp_1} = \frac{iA(v_1^2 - v_4^2)}{2\pi L'}\left[g(\eta_0^1)\left(t - \frac{y\eta_0^1}{v_1^2\sqrt{\left(\frac{\eta_0^1}{v_1}\right)^2 + 1}} \right)^{-1} \Delta_0^{-1}(\eta_0^1) \right.$$

$$\left. - g(\overline{\eta_0^1})\left(t - \frac{y\overline{\eta_0^1}}{v_1^2\sqrt{\left(\frac{\overline{\eta_0^1}}{v_1}\right)^2 + 1}} \right)^{-1} \Delta_0^{-1}(\overline{\eta_0^1}) \right]. \tag{29}$$

Further considerations will essentially depend on the sign of $\Upsilon(\xi_{st})$. We see that the case of $\Upsilon(\xi st) < 0$ corresponds to the situation when the wave P_1 has not yet reached the point (x, y). If $\Upsilon(\xi st) = 0$, then the point (x, y) is located on the wave front P_1. And in the case of $\Upsilon(\xi st) > 0$ we are inside the wave front.

- The case $\Upsilon(\xi st) > 0$.

We set $\Psi(\eta_{st}) = i\Upsilon(\xi_{st})$. Then Ψ_-^1 has no roots on the imaginary axis from ξ_{st} to $+\infty$, that is, the unique root inside the closed contour has a positive real part. Thus, the residue for this root is also not purely imaginary and in the sum with the complex conjugate value (the residue for Ψ_+^1) will give horizontal component of the displacements of the longitudinal fast wave P_1:

$$u_{xp_1} = \frac{-A(v_1^2 - v_4^2)}{\pi L^{'}} \mathrm{Im}\left[g(\eta_0^1)\left(t - \frac{y\eta_0^1}{v_1^2\sqrt{\left(\frac{\eta_0^1}{v_1}\right)^2 + 1}} \right)^{-1} \Delta_0^{-1}(\eta_0^1) \right]. \tag{30}$$

- The case $\Upsilon(\xi st) < 0$.

Note that $\Upsilon(\xi_{st}) > 0$ as $\eta \to +\infty$, i. e. $\Psi(\eta)$ has a root on the imaginary axis belonging to the interval $(\xi_{st}, +\infty)$. The root to the right of the contour is unique, hence

$$\mathrm{Re}\eta_0^1 = \mathrm{Re}\frac{ixt + y\sqrt{t^2 - \frac{x^2 + y^2}{v_1^2}}}{t^2 - \frac{y^2}{v_1^2}} = 0.$$

Such an equality is possible only for $v_1 t < \sqrt{x^2 + y^2}$, i. e. the front of the wave P_1 (a semicircle centered at the origin and radius $v_1 t$) has not yet reached the point (x, y). For a purely imaginary root η_0^1, the residues for $\Psi_\pm^1$ are real and equal. Where does it follow that

$$u_{xp_1} = 0.$$

- The case $\Upsilon(\xi st) = 0$.

One can see that η_0^1 is the root of the function $\Psi(\eta)$ passes from the imaginary axis to the right half-plane when

$$t^2 = \frac{x^2 + y^2}{v_1^2}.$$

At this point, u_{xp1} becomes nonzero. This case corresponds to the position of the point (x, y) on the wave front of the wave P_1.

The derivation of explicit expressions for the displacement components u_{xp_2}, u_{xs} for the slow longitudinal P_2 and the transverse S waves is much the same as u_{xp_1}. However, the difference from the case with the fast longitudinal wave P_1 is as follows: to change the order of integration with respect to η and k, the contour should be deformed by encircling the cuts beginning at the points $\pm v_j$, $j = 1,2,3$ and that the case $\Upsilon(\xi st) < 0$, in addition to the situation when the front of the volume wave has not yet reached the point (x, y), corresponds to *the head waves*. Let the material parameters of the Biot medium be such that $v_2 > v_3$. Consider for $t > \dfrac{y}{v_2}$

$$u_{xp_2} = \frac{A}{\pi L'} \int_0^{+\infty} \frac{\sin kx}{2\pi i} dk \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{(v_4^2 - v_2^2) g e^{k(-\alpha_2 y + t\eta)})}{\Delta_0} d\eta. \tag{31}$$

We make the substitution $v_1 \to v_2$ in the formula (24). Let ξ_{st} be the root of $\Upsilon'(\xi)$. Now, following the derivations for u_{xp_1}, we draw the integration contour, where $\mathrm{Re}\Psi(\eta) < 0$. As follows from our considerations, $i\xi_{st}$ lies on the imaginary axis higher than iv_2, the point at which one of the cuts begins. As a consequence, $i\xi_{st}$ can be located between points iv_2 and iv_1. In this case, we draw the contour, encircling the corresponding cuts at the points $\pm iv_1$ (see Fig. 2). Similarly to the expression (26), we get

$$u_{xp_2} = -\frac{1}{4\pi^2 L'} \int_{C^2} \frac{(v_4^2 - v_2^2) g}{\Delta_0} \left(\frac{1}{\Psi_-^2} - \frac{1}{\Psi_+^2} \right) d\eta, \tag{32}$$

where

$$\Psi_\pm^2 = \pm ix + t\eta - \alpha_2 y, \tag{33}$$

and C^2 is the contour shown in Fig. 2.

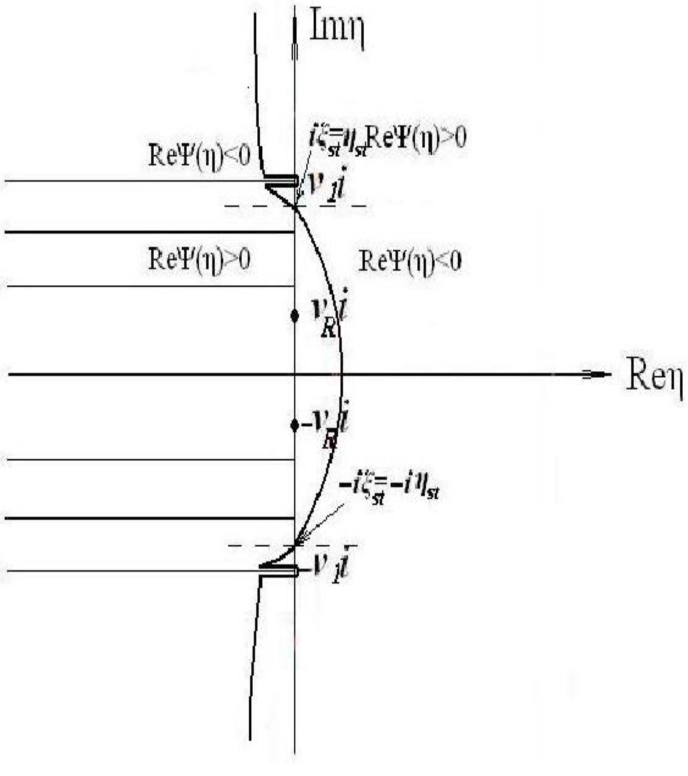

Figure 2. Integration contour C^2; the case of P_2 wave

Using the residue theorem, we arrive at

$$u_{xp_1} = \frac{iA(v_4^2 - v_2^2)}{2\pi L'}\left[g(\eta_0^2)\left(t - \frac{y\eta_0^2}{v_2^2\sqrt{\left(\frac{\eta_0^2}{v_2}\right)^2 + 1}} \right)^{-1} \Delta_0^{-1}(\eta_0^2) \right.$$

$$\left. - g(\overline{\eta_0^2})\left(t - \frac{y\overline{\eta_0^2}}{v_2^2\sqrt{\left(\frac{\overline{\eta_0^2}}{v_2}\right)^2 + 1}} \right)^{-1} \Delta_0^{-1}(\overline{\eta_0^2}) \right], \quad (34)$$

where

$$\eta_0^2 = \frac{ixt + y\sqrt{t^2 - \frac{x^2+y^2}{v_2^2}}}{t^2 - \frac{y^2}{v_2^2}} \quad (35)$$

is the unique root of $\Psi(\eta) = -ix + t\eta - y\sqrt{\left(\frac{\eta}{v_2}\right)^2 + 1}$ to the right of the contour of integration. Note that if $\Psi_-^2(\eta_0^2) = 0$, then $\Psi_+^2(\overline{\eta_0^2}) = 0$. Further considerations again depend on the sign of $\Upsilon(\xi_{st})$, but do not fully correspond to the case of the fast longitudinal wave P_1. We will see that the case $\Upsilon(\xi_{st}) < 0$, in addition to the situation when the slow longitudinal wave P_2 has not yet reached the point (x, y), also corresponds to the head wave P_1P_2.

- The case $\Upsilon(\xi st) > 0$.

Let $\Psi(\eta_{st}) = i\Upsilon(\xi_{st})$. Then Ψ_-^2 has no roots on the imaginary axis from ξ_{st} to $+\infty$, i. e. the unique root inside the closed contour has a positive real part. Thus, the residue for this root is also not purely imaginary and in the sum with the complex conjugate value (the residue for Ψ_+^2) will give horizontal component of the displacements of the longitudinal slow wave P_2:

$$u_{xp_2} = \frac{-A(v_4^2 - v_2^2)}{\pi L'} \mathrm{Im}\left[g(\eta_0^2)\left(t - \frac{y\eta_0^2}{v_2^2\sqrt{\left(\frac{\eta_0^2}{v_2}\right)^2 + 1}} \right)^{-1} \Delta_0^{-1}(\eta_0^2) \right]. \quad (36)$$

- The case $\Upsilon(\xi st) = 0$.

One can see that η_0^2 is the root of the function $\Psi(\eta)$ passes from the imaginary axis to the right half-plane when

$$t^2 = \frac{x^2 + y^2}{v_2^2}.$$

At this point, u_{xp2} becomes nonzero. This case corresponds to the position of the point (x, y) on the wave front of the wave P_2.

- The case $\Upsilon(\xi st) < 0$.

The considerations in this case will be most significantly different from the case for u_{xp_2}. Here we show how the head wave P_1P_2 arises. Let ξ_0^2 be the root of the function

$$\Upsilon(\xi_0^2) = x + \xi_0^2 t - y\sqrt{\left(\frac{\xi_0^2}{v_2}\right)^2 - 1} = 0. \quad (37)$$

There are three possible locations for ξ_0^2 on the imaginary axis relative to the cuts and ξ_{st}:

- $v^1 < \xi_0^2$. Since $\xi_{st} > v_2$, the point $i\xi_{st}$ lies on the imaginary axis above all the cuts drawn from the points iv_j, $j = 1,2,3$. The point $i\xi_0^2$ is even higher, i.e. $\sqrt{\left(\frac{\eta_0^2}{v_j}\right)^2 + 1}$, $j = 1,2,3$ are purely imaginary expressions. The residues for $\Psi_\pm^2$ will be real and equal. Where does it follow that

$$u_{xp_2} = 0.$$

So we are in front of the wave front of the head wave P_1P_2.

- $v^2 < \xi_{st} < \xi_0^2 < v_1$. The radicals $\sqrt{\left(\frac{\eta_0^2}{v_j}\right)^2 + 1}$, $j = 2,3$ remain purely imaginary, but

$$\sqrt{\left(\frac{\eta_0^2}{v_1}\right)^2 + 1} = \sqrt{1 - \left(\frac{\eta_0^2}{v_1}\right)^2} > 0.$$

This means that $u_{xp_2} \neq 0$. This situation corresponds to the case when the point (x, y) is inside the wave front of the head wave P_1P_2, but outside the wave front of the wave P_2.

- $v^2 < \xi_{st} < \xi_0^2 = v_1$. This situation is transitional between the above considered and corresponds to the front of the head wave P_1P_2. It is easy to show that the wave front of the head wave P_1P_2 has equation

$$\pm x + v_1 t - y\sqrt{\left(\frac{v_1}{v_2}\right)^2 - 1} = 0.$$

Carrying out analogous calculations for u_{xs}, u_y, w_x, w_y and, summing up the results obtained, we obtain the formulas

$$u_x = u_{xp1} + u_{xp2} + u_{xs},\ u_y = u_{yp1} + u_{yp2} + u_{ys}, \quad (38)$$

$$w_x = w_{xp1} + w_{xp2} + w_{xs},\ w_y = w_{yp1} + w_{yp2} + w_{ys}, \quad (39)$$

Here,

$$u_{xp_1} = \frac{-A(v_1^2 - v_4^2)}{\pi L'} \operatorname{Im}\left[g(\eta_0^1)\left(t - \frac{y\eta_0^1}{v_1^2\sqrt{\left(\frac{\eta_0^1}{v_1}\right)^2 + 1}} \right)^{-1} \Delta_0^{-1}(\eta_0^1) \right] \tag{40}$$

is the horizontal component of the displacements of the fast longitudinal wave P_1,

$$u_{xp_2} = \frac{-A(v_4^2 - v_2^2)}{\pi L'} \operatorname{Im}\left[g(\eta_0^2)\left(t - \frac{y\eta_0^2}{v_2^2\sqrt{\left(\frac{\eta_0^2}{v_2}\right)^2 + 1}} \right)^{-1} \Delta_0^{-1}(\eta_0^2) \right] \tag{41}$$

is the horizontal component of the displacements of the slow longitudinal wave P_2,

$$u_{xs} = \frac{2A}{\pi L'} \operatorname{Im}\left[\frac{\sqrt{\left(\frac{\eta_0^3}{v_3}\right)^2 + 1}\left(\sqrt{\left(\frac{\eta_0^3}{v_1}\right)^2 + 1}(v_1^2 - v_4^2) + \sqrt{\left(\frac{\eta_0^3}{v_2}\right)^2 + 1}(v_4^2 - v_2^2) \right)}{\left(t - \frac{y\eta_0^3}{v_3^2\sqrt{\left(\frac{\eta_0^3}{v_3}\right)^2 + 1}} \right)\Delta_0(\eta_0^3)} \right] \tag{42}$$

is the horizontal component of the displacements of the transverse wave S,

$$u_{yp_1} = \frac{-A(v_1^2 - v_4^2)}{\pi L'} \operatorname{Re}\left[g(\eta_0^1)\alpha_1(\eta_0^1)\left(t - \frac{y\eta_0^1}{v_1^2\sqrt{\left(\frac{\eta_0^1}{v_1}\right)^2 + 1}} \right)^{-1} \Delta_0^{-1}(\eta_0^1) \right] \tag{43}$$

is the vertical component of the displacements of the longitudinal fast wave P_1,

$$u_{yp_2} = \frac{-A(v_4^2 - v_2^2)}{\pi L'} \operatorname{Re}\left[g(\eta_0^2)\alpha_2(\eta_0^2)\left(t - \frac{y\eta_0^2}{v_2^2\sqrt{\left(\frac{\eta_0^2}{v_2}\right)^2 + 1}} \right)^{-1} \Delta_0^{-1}(\eta_0^2) \right] \tag{44}$$

is the vertical component of the displacements of the longitudinal slow wave P_2,

$$u_{ys} = \frac{2A}{\pi L'} \mathrm{Re}\left[\frac{\left(\sqrt{\left(\frac{\eta_0^3}{v_1}\right)^2 + 1}(v_1^2 - v_4^2) + \sqrt{\left(\frac{\eta_0^3}{v_2}\right)^2 + 1}(v_4^2 - v_2^2) \right)}{\left(t - \frac{y\eta_0^3}{v_3^2 \sqrt{\left(\frac{\eta_0^3}{v_3}\right)^2 + 1}} \right) \Delta_0(\eta_0^3)} \right] \tag{45}$$

is the vertical component of the displacements of the transverse wave S, where

$$\eta_0^j = \frac{ixt + y\sqrt{t^2 - \frac{x^2 + y^2}{v_j^2}}}{t^2 - \frac{y^2}{v_j^2}} \ (j = 1,2,3). \tag{46}$$

The relative displacements in the fluid phase are expressed in terms of displacements in the elastic phase in accordance with the formulas

$$w_{xp_1} = B_1 u_{xp_1},\ w_{xp_2} = B_2 u_{xp_2},\ w_{xs} = -\frac{\rho_f}{m} u_{xs}, \tag{47}$$

$$w_{yp_1} = B_1 u_{yp_1},\ w_{yp_2} = B_2 u_{yp_2},\ w_{ys} = -\frac{\rho_f}{m} u_{ys}. \tag{48}$$

From the exact solution (38) – (48) we can single out the expression that corresponds to the surface Rayleigh wave. In the first approximation as $y \to 0$, the components of the displacements of the Rayleigh wave have the form

$$u_x^R \approx \frac{A(v_1^2 - v_4^2)}{\pi L'} \left[\frac{\frac{y}{t}\frac{1}{t}\frac{c_2}{c_1^2}}{(\frac{x}{t} \pm v_R)^2 + \left(\frac{c_2}{c_1}\right)^2 \left(\frac{y}{t}\right)^2} \right] + \frac{A(v_4^2 - v_2^2)}{\pi L'} \left[\frac{\frac{y}{t}\frac{1}{t}\frac{c_4}{c_3^2}}{(\frac{x}{t} \pm v_R)^2 + \left(\frac{c_4}{c_3}\right)^2 \left(\frac{y}{t}\right)^2} \right]$$
$$- \frac{2A}{\pi L'} \left[\frac{\frac{y}{t}\frac{1}{t}\frac{c_6}{c_5^2}}{(\frac{x}{t} \pm v_R)^2 + \left(\frac{c_6}{c_5}\right)^2 \left(\frac{y}{t}\right)^2} \right], \tag{49}$$

$$u_y^R \approx \frac{A(v_1^2 - v_4^2)}{\pi L'} \left[\frac{(\frac{x}{t} \pm v_R)\frac{1}{t}\frac{\tilde{c}_2}{\tilde{c}_1^2}}{(\frac{x}{t} \pm v_R)^2 + \left(\frac{\tilde{c}_2}{\tilde{c}_1}\right)^2 \left(\frac{y}{t}\right)^2} \right] + \frac{A(v_4^2 - v_2^2)}{\pi L'} \left[\frac{(\frac{x}{t} \pm v_R)\frac{1}{t}\frac{\tilde{c}_4}{\tilde{c}_3^2}}{(\frac{x}{t} \pm v_R)^2 + \left(\frac{\tilde{c}_4}{\tilde{c}_3}\right)^2 \left(\frac{y}{t}\right)^2} \right]$$

$$-\frac{2A}{\pi L'}\left[\frac{(\frac{x}{t}\pm v_R)\frac{1}{t}\frac{\tilde{c}_6}{\tilde{c}_5^{\,2}}}{(\frac{x}{t}\pm v_R)^2+\left(\frac{\tilde{c}_6}{\tilde{c}_5}\right)^2\left(\frac{y}{t}\right)^2}\right], \tag{50}$$

$$w_x^R = B_1 u_{xp1}^R + B_2 u_{xp2}^R - \frac{\rho_f}{m} u_{xs}^R, \tag{51}$$

$$w_y^R = B_1 u_{yp1}^R + B_2 u_{yp2}^R - \frac{\rho_f}{m} u_{ys}^R, \tag{52}$$

where $c_l(v_j, v_R), \tilde{c}_l(v_j, v_R) = const, i = 1,2,3,4,5,6, j = 1,2,3,4,$ are parameters with dimension of velocity; $u_{xp1}^R, u_{xp2}^R, u_{xs}^R$ are the first, second, and third summands in (49), $u_{yp1}^R, u_{yp2}^R, u_{ys}^R$ are the first, second, and third summands in (50).

Analyzing the solution of the problem obtained above, we have different analytic expressions in different domains. The passage from some expressions to others determines wave front sets, i.e., lines on which there are singularities. In the case of a point source that is situated on the boundary of the porous medium, in addition to the volume spherical waves P_1, P_2, S and to the surface Rayleigh wave, three head waves P_1P_2, P_1S, and $P_2S(v_2 > v_3)$ or $SP_2(v_3 > v_2)$ propagate. The picture of wave front sets is given in Fig. 3.

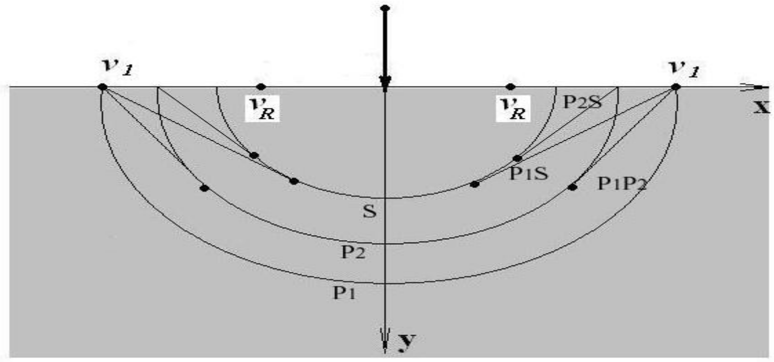

Figure 3. Wave front sets

In conclusion, we note that another approach to this problem is possible on the base of papers (Smirnoff& Soboleff, 1932). One can show that the formulas obtained by the Petrashen's method, coincide exactly with the formulas that are derived with the method of complex solutions (see (Petrashen *et al.*, 1950; Smirnoff & Soboleff, 1932).

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